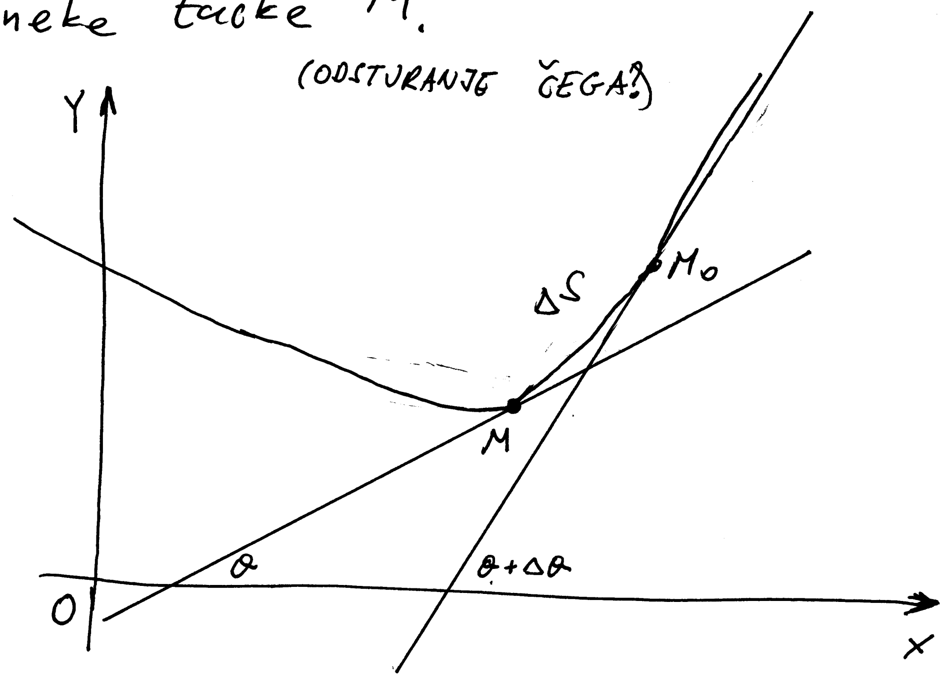
Zakrivljenost i torzija krive

Zakrivljenost ili krivina krive u ravni je veličina koja karakteriše stepen njenog odstupanja od prave u okolini neke tačke M.

(ODSTUPANJE ČEGA?)



Pravac krive u tački M se može okarakterisati uglom α koji gradi tangenta na krivu u tački M s osom Ox (vidi sliku). Brzina mijenjanja ugla α pri ravnomjernom kretanju tačke M po krivoj naziva se krivina krive u tački M

Torzija krive je brzina obrtanja oskulatorne ravni krive u tački A ako se tačka A kreće jednako (ravnomjerno) po krivoj brzinom jednakom jedinici.

Na osnovu ove definicije, šta znači ako je torzija uvijek jednaka nuli.

Date definicije krivine i torzije su opisne definicije.

Krivinu krive ćemo označavati sa K a poluprečnik krivine sa $R = \frac{1}{K}$. Torziju ćemo označavati sa $\frac{1}{T}$ a poluprečnik torzije sa $|T|$.

(#) Odrediti jedinične vektore tangente, glavne normale i binormale krive

$$x = e^t \cos t, \quad y = e^t \sin t, \quad z = e^t.$$

Zatim naći krivinu i torziju date krive.

R:

Vektore tangente, binormale i normale krive određujemo formulama $\vec{t} = \dot{\vec{r}}$, $\vec{b} = \dot{\vec{r}} \times \ddot{\vec{r}}$, $\vec{n} = \vec{b} \times \vec{t}$.

$$\vec{r} = (e^t \cos t, \; e^t \sin t, \; e^t)$$

$$\dot{\vec{r}} = (e^t \cos t - e^t \sin t, \; e^t \sin t + e^t \cos t, \; e^t)$$

$$\ddot{\vec{r}} = (\underline{e^t \cos t} - e^t \sin t - e^t \sin t \underline{- e^t \cos t}, \; \underline{e^t \sin t} + e^t \cos t + e^t \cos t \underline{- e^t \sin t}, \; e^t)$$

$$= (-2e^t \sin t, \; 2e^t \cos t, \; e^t)$$

$$\dddot{\vec{r}} = e^t(-2\sin t - 2\cos t, \; 2\cos t - 2\sin t, \; 1)$$

$$\vec{t} = e^t(\cos t - \sin t, \; \sin t + \cos t, \; 1)$$

$$|\vec{t}|^2 = e^{2t}(\underline{\cos^2 t} - \cancel{2\cos t \sin t} + \underline{\sin^2 t} + \underline{\sin^2 t} + \cancel{2\sin t \cos t} + \underline{\cos^2 t} + 1) =$$

$$= 3e^{2t} \Rightarrow |\vec{t}| = e^t\sqrt{3}$$

Jedinični vektor tangente je

$$\vec{t}_0 = \frac{\vec{t}}{|\vec{t}|} = \frac{1}{\sqrt{3}}(\cos t - \sin t, \; \sin t + \cos t, \; 1)$$ jedinični vektor tangente

$$\vec{b} = \dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ e^t \cos t - e^t \sin t & e^t \sin t + e^t \cos t & e^t \\ -2e^t \sin t & 2e^t \cos t & e^t \end{vmatrix} =$$

$$= e^t \cdot e^t \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos t - \sin t & \sin t + \cos t & 1 \\ -2\sin t & 2\cos t & 1 \end{vmatrix} =$$

$$= e^{2t}(\sin t + \cos t - 2\cos t, \; -(\cos t - \sin t + 2\sin t), \; 2\cos^2 t - \cancel{2\sin t \cos t} + 2\sin^2 t + \cancel{2\sin t \cos t})$$

$$\vec{b} = e^{2t}(\sin t - \cos t, -\sin t - \cos t, 2)$$

$$|\vec{b}|^2 = e^{4t}\cdot(\sin^2 t - \cancel{2\sin t\cos t} + \cos^2 t + \sin^2 t + \cancel{2\sin t\cos t} + \cos^2 t + 4)$$

$$|\vec{b}| = e^{2t}\sqrt{6}$$

$$\vec{b}_0 = \frac{\vec{b}}{|\vec{b}|} = \frac{1}{\sqrt{6}}(\sin t - \cos t, -(\sin t + \cos t), 2)$$ jedinični vektor binormale

$$\vec{n}_0 = \vec{b}_0 \times \vec{t}_0 = \frac{1}{\sqrt{6}}\cdot\frac{1}{\sqrt{3}}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \sin t - \cos t & -\sin t - \cos t & 2 \\ \cos t - \sin t & \sin t + \cos t & 1 \end{vmatrix} =$$

$$= \frac{1}{\sqrt{18}}\big(-\sin t - \cos t - 2\sin t - 2\cos t, -(\sin t - \cos t - 2\cos t + 2\sin t), \sin^2 t - \cos^2 t + \cos^2 t - \sin^2 t\big) =$$

$$= \frac{1}{3\sqrt{2}}\big((-3)(\sin t + \cos t), (-3)\cdot(\sin t - \cos t), 0\big) =$$

$$\vec{n}_0 = \frac{1}{\sqrt{2}}\big((-1)(\sin t + \cos t), -\sin t + \cos t, 0\big)$$ jedinični vektor glavne normale

Krivinu krive možemo izračunati po formuli $K = \frac{1}{R}$ gde je R poluprečnik krivine $R = \frac{|\dot{\vec{r}}|^3}{|\vec{b}|}$

$$R = \frac{(e^t\sqrt{3})^3}{e^{2t}\sqrt{6}} = \frac{e^{3t}\,3\sqrt{3}}{e^{2t}\sqrt{6}} = 3e^t\sqrt{\frac{1}{2}} \Rightarrow K = \frac{\sqrt{2}}{3e^t}$$ tražena krivina krive

Torziju možemo izračunati po formuli $-\tau = \frac{1}{T} = \frac{\dddot{\vec{r}}\cdot\vec{b}}{|\vec{b}|^2}$

$$\dddot{\vec{r}} = e^t(-2\sin t - 2\cos t, \overbrace{2\cos t - 2\sin t}^{2(\cos t - \sin t)}, 1)$$

$$\vec{b} = e^{2t}(\sin t - \cos t, \underbrace{-\sin t - \cos t}_{(-1)(\cos t + \sin t)}, 2)$$

$$\dddot{\vec{r}}\cdot\vec{b} = e^{3t}\big((-2)(\sin^2 t - \cos^2 t) - 2(\cos^2 t - \sin^2 t) + 2\big) = 2e^{3t}$$

$$-\tau = \frac{2e^{3t}}{e^{4t}\sqrt{6}} = \sqrt{\frac{4}{6}}\cdot\frac{1}{e^t} = \frac{\sqrt{2}}{e^t\sqrt{3}}$$ tražena torzija krive ← GREŠKA

(#) Pokazati da su kod krive

$$x = \operatorname{ch} z \qquad y = \operatorname{sh} z$$

radijus krive i torzije (R i T) jednaki.

Rj. Ako uvedemo smjenu $z = t$ datu krivu možemo napisati u obliku

$$C \curvearrowright \vec{r} : \begin{cases} x = \operatorname{ch} t \\ y = \operatorname{sh} t \\ z = t \end{cases}$$

Tada je

$$\dot{\vec{r}} = (\operatorname{sh} t, \operatorname{ch} t, 1)$$
$$\ddot{\vec{r}} = (\operatorname{ch} t, \operatorname{sh} t, 0)$$
$$\dddot{\vec{r}} = (\operatorname{sh} t, \operatorname{ch} t, 0)$$

Neka je $\dot{\vec{r}} = \dot{\vec{r}}(\dot{x}, \dot{y}, \dot{z})$
$\ddot{\vec{r}} = \ddot{\vec{r}}(\ddot{x}, \ddot{y}, \ddot{z})$
$\dddot{\vec{r}} = \dddot{\vec{r}}(\dddot{x}, \dddot{y}, \dddot{z})$

Tada je

$$R^2 = \frac{(\dot{\vec{r}}^{\,2})^3}{[\dot{\vec{r}} \times \ddot{\vec{r}}]^2}, \quad a \quad T = \frac{-[\dot{\vec{r}} \times \ddot{\vec{r}}]^2}{\dot{\vec{r}}(\ddot{\vec{r}} \times \dddot{\vec{r}})}$$

$$\dot{\vec{r}}^{\,2} = \dot{\vec{r}} \cdot \dot{\vec{r}} = \operatorname{sh}^2 t + \operatorname{ch}^2 t + 1 = 2\operatorname{ch}^2 t \qquad \boxed{\operatorname{sh}^2 t + 1 = \operatorname{ch}^2 t}$$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \operatorname{sh} t & \operatorname{ch} t & 1 \\ \operatorname{ch} t & \operatorname{sh} t & 0 \end{vmatrix} = (-\operatorname{sh} t, \operatorname{ch} t, \operatorname{sh}^2 t - \operatorname{ch}^2 t)$$
$$= (-\operatorname{sh} t, \operatorname{ch} t, -1)$$

$$(\dot{\vec{r}} \times \ddot{\vec{r}})^2 = \operatorname{sh}^2 t + \operatorname{ch}^2 t + 1 = 2\operatorname{ch}^2 t$$

$$R^2 = \frac{(2\operatorname{ch}^2 t)^3}{2\operatorname{ch}^2 t} = (2\operatorname{ch}^2 t)^2 \quad \Rightarrow \quad R = |2\operatorname{ch}^2 t| = 2\operatorname{ch}^2 t$$

$$\dot{\vec{r}}(\ddot{\vec{r}} \times \dddot{\vec{r}}) = \begin{vmatrix} \operatorname{sh} t & \operatorname{ch} t & 1 \\ \operatorname{ch} t & \operatorname{sh} t & 0 \\ \operatorname{sh} t & \operatorname{ch} t & 0 \end{vmatrix} = -1 \underbrace{(\operatorname{ch}^2 t - \operatorname{sh}^2 t)}_{-1} = +1$$

$$T = +\frac{2\operatorname{ch}^2 t}{+1} = 2\operatorname{ch}^2 t$$

$$R = |T| = 2\operatorname{ch}^2 t$$

traženo rješenje

(#) Naći poluprečnik torzije $|T|$ za krivu

$$\vec{r} = \cos t\,\vec{i} + \sin t\,\vec{j} + \mathrm{sh}\,t\,\vec{k}.$$

Rj. Poluprečnik torzije $|T|$ možemo naći po formuli

$$T = -\frac{[\dot{\vec{r}} \times \ddot{\vec{r}}]^2}{\dot{\vec{r}}\,[\ddot{\vec{r}} \times \dddot{\vec{r}}]}$$

$$\dot{\vec{r}} = (-\sin t,\ \cos t,\ \mathrm{ch}\,t)$$

$$\ddot{\vec{r}} = (-\cos t,\ -\sin t,\ \mathrm{sh}\,t)$$

$$\dddot{\vec{r}} = (\sin t,\ -\cos t,\ \mathrm{ch}\,t)$$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin t & \cos t & \mathrm{ch}\,t \\ -\cos t & -\sin t & \mathrm{sh}\,t \end{vmatrix} = (\cos t\,\mathrm{sh}\,t + \sin t\,\mathrm{ch}\,t,\ \sin t\,\mathrm{sh}\,t - \cos t\,\mathrm{ch}\,t,\ 1)$$

to je

$$[\dot{\vec{r}} \times \ddot{\vec{r}}]^2 = \cos^2 t\,\mathrm{sh}^2 t + 2\sin t\,\cancel{\cos t\,\mathrm{sh}\,t}\,\mathrm{ch}\,t + \sin^2 t\,\mathrm{ch}^2 t + \sin^2 t\,\mathrm{sh}^2 t - 2\sin t\,\cancel{\cos t\,\mathrm{sh}\,t}\,\mathrm{ch}\,t + \cos^2 t\,\mathrm{ch}^2 t + 1 =$$

$$= \mathrm{sh}^2 t + \mathrm{ch}^2 t + 1 = \mathrm{ch}^2 t + \left(\frac{e^t - e^{-t}}{2}\right)^2 + 1 = \mathrm{ch}^2 t + \frac{e^{2t} - 2 + e^{-2t}}{4} + 1$$

$$= \mathrm{ch}^2 t + \frac{e^{2t} + 2 + e^{-2t}}{4} = 2\,\mathrm{ch}^2 t$$

$$\dot{\vec{r}}\,[\ddot{\vec{r}} \times \dddot{\vec{r}}] = \begin{vmatrix} -\sin t & \cos t & \mathrm{ch}\,t \\ -\cos t & -\sin t & \mathrm{sh}\,t \\ \sin t & -\cos t & \mathrm{ch}\,t \end{vmatrix} \overset{I_v + III_v}{=\!=\!=} \begin{vmatrix} 0 & 0 & 2\,\mathrm{ch}\,t \\ -\cos t & -\sin t & \mathrm{sh}\,t \\ \sin t & -\cos t & \mathrm{ch}\,t \end{vmatrix} = 2\,\mathrm{ch}\,t$$

$$|T| = \left| -\frac{2\,\mathrm{ch}^2 t}{2\,\mathrm{ch}\,t} \right| = \mathrm{ch}\,t = \frac{e^t + e^{-t}}{2}$$

traženi poluprečnik torzije

(#) Naći radijus krivine i krivinu krive

$$C: \begin{cases} x = \sin z - z\cos z \\ y = \cos z + z\sin z \end{cases}$$

u proizvoljnoj tački.

Rj. Kao parametar stavimo $z = t$. Tada

$$\vec{r} = (\sin t - t\cos t,\ \cos t + t\sin t,\ t)$$

Krivina krive K je data izrazom $K = \dfrac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3}$,

a poluprečnik krivine je $R = \dfrac{1}{K}$.

$$\dot{\vec{r}} = (\cos t - \cos t + t\sin t,\ -\sin t + \sin t + t\cos t,\ 1)$$
$$= (t\sin t,\ t\cos t,\ 1)$$

$$\ddot{\vec{r}} = (\sin t + t\cos t,\ \cos t - t\sin t,\ 0)$$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t\sin t & t\cos t & 1 \\ \sin t + t\cos t & \cos t - t\sin t & 0 \end{vmatrix} = (t\sin t - \cos t,\ \sin t + t\cos t,\ \underline{t\sin t\cos t} - t^2\sin^2 t \underline{- t\sin t\cos t} - t^2\cos^2 t)$$

$$= (t\sin t - \cos t,\ \sin t + t\cos t,\ -t^2)$$

$$|\dot{\vec{r}} \times \ddot{\vec{r}}|^2 = (\dot{\vec{r}} \times \ddot{\vec{r}})^2 = t^2\sin^2 t \underline{\cancel{- 2t\sin t\cos t}} + \underbrace{\cos^2 t + \sin^2 t}_{=1} \underline{\cancel{+ 2t\sin t\cos t}} + t^2\cos^2 t$$
$$+ t^4 = 1 + t^2 + t^4$$

$$|\dot{\vec{r}}|^2 = t^2\sin^2 t + t^2\cos^2 t + 1 = t^2 + 1$$

$$|\dot{\vec{r}}| = \sqrt{t^2 + 1}$$

$$K = \frac{\sqrt{1 + t^2 + t^4}}{\sqrt{(t^2+1)^3}} \quad ; \quad R = \frac{\sqrt{(t^2+1)^3}}{\sqrt{1 + t^2 + t^4}}$$

(#) Napisati jednačinu skupa tačaka u kojima tangente zavojnice $\vec{r}=(a\cos t, a\sin t, bt)$ prodiru ravan $z=0$. Odrediti zakrivljenost dobijene krive.

Rj: Pronađimo prvo jednačinu tangente na zavojnicu u proizvoljnoj tački $M(t)$.

$$\frac{d\vec{r}}{dt}=(-a\sin t, a\cos t, b)$$

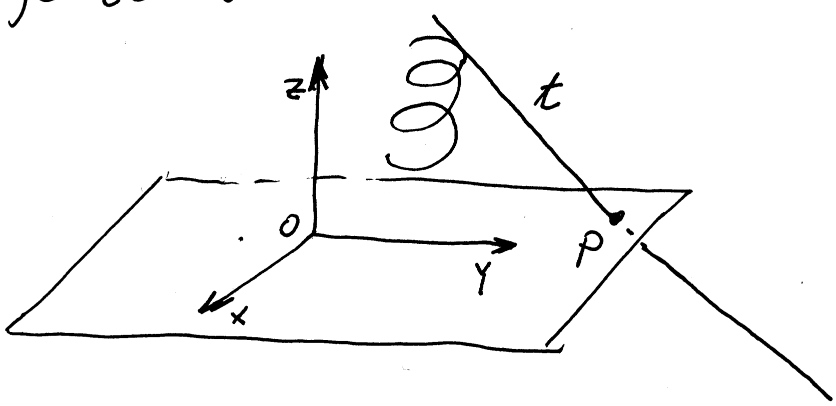
Ako uzmemo tačku $M(a\cos t, a\sin t, bt)$ imamo sljedeću jednačinu tangente

$$t: \quad \frac{x-a\cos t}{-a\sin t}=\frac{y-a\sin t}{a\cos t}=\frac{z-bt}{b}$$

U parametarskom obliku, gdje je u parametar, imamo

$$t: \begin{cases} x=a\cos t-au\sin t \\ y=a\sin t+au\cos t \\ z=bt+ub \\ \quad u\in\mathbb{R} \end{cases}$$

u je tekuća koordinata za tačke tangente



Prodor P tangente $\vec{t}$ sa ravni $z=0$ dobija se za $bt+ub=0$, tj. za $u=-t$.

Dakle koordinate tačke prodora P su

$$(a\cos t+at\sin t, a\sin t-at\cos t, 0)$$

Za različitu vrijednost parametra t drugu tačku na zavojnici, drugačiju tangentu na zavojnicu a samim time i drugačiju tačku prodora.

Skup tačaka u kojima tangente zavojnice $\vec{r}$ prodiru ravan $z=0$ formiraju sljedeću krivu

$$\vec{r}^*: \begin{cases} x = a\cos t + at\sin t \\ y = a\sin t - at\cos t \\ z = 0 \\ -\infty < t < +\infty \end{cases}$$

Sad trebamo odrediti zakrivljenost dobijene krive

$$\vec{t}^* = \dot{\vec{r}}^* = (-\cancel{a\sin t} + \cancel{a\sin t} + at\cos t,\ \cancel{a\cos t} - \cancel{a\cos t} + at\sin t,\ 0)$$

Krivina K je određena relacijom $K = \frac{1}{R}$ gdje je R poluprečnik krivine određen relacijom $R = \frac{|\vec{t}|^3}{|\vec{b}|}$.

$$\vec{b}^* = \dot{\vec{r}}^* \times \ddot{\vec{r}}^* = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ at\cos t & at\sin t & 0 \\ a\cos t - at\sin t & a\sin t + at\cos t & 0 \end{vmatrix} \overset{(**)}{=}$$

$$\ddot{\vec{r}}^* = (a\cos t - at\sin t,\ a\sin t + at\cos t,\ 0)$$

$$\overset{(**)}{=} a^2 \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t\cos t & t\sin t & 0 \\ \cos t - t\sin t & \sin t + t\cos t & 0 \end{vmatrix} = a^2(0, 0, \underline{t\sin t\cos t} + t^2\cos^2 t \underline{- t\sin t\cos t} + t^2\sin^2 t)$$

$$= a^2(0, 0, t^2) \qquad\qquad = a^2(0, 0, t^2)$$

$$|\vec{t}^*| = \sqrt{a^2t^2\cos^2 t + a^2t^2\sin^2 t + 0} = at$$

$$|\vec{t}^*|^3 = a^3t^3$$

$$|\vec{b}^*| = a^2\sqrt{0 + 0 + t^4} = a^2t^2$$

$$R = \frac{a^3t^3}{a^2t^2} = at \qquad K = \frac{1}{at}$$ tražena zakrivljenost date krive

Izračunati torziju krive $\vec{r} = a(1-\cos t, \sin 2t, 2\cos t)$ u proizvoljnoj tački. Odrediti jednačinu ravni kojoj kriva pripada.

R.: Torziju krive možemo izračunati po formuli

$$\frac{1}{T} = -\frac{\dot{\vec{r}}\,[\ddot{\vec{r}} \times \dddot{\vec{r}}]}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2}$$

Kako je

$$\dot{\vec{r}}\,[\ddot{\vec{r}} \times \dddot{\vec{r}}] = a^3 \begin{vmatrix} \sin t & 2\cos 2t & -2\sin t \\ \cos t & -4\sin 2t & -2\cos t \\ -\sin t & -8\cos 2t & 2\sin t \end{vmatrix} \overset{III_k + I_k \cdot 2}{=\!=\!=} 0$$

To je torzija $\frac{1}{T} = 0$

Ako je torzija $\frac{1}{T} = 0$ u svakoj tački krive, onda kriva leži u ravni. Ta ravan u ovom slučaju ima jednačinu $Ax + By + Cz + D = 0$.

U našem slučaju

$$A(a - a\cos t) + Ba\sin 2t + C2a\cos t + D = 0 \qquad /:a$$

$$(A+D) + (2C - A)\cos t + B\sin 2t = 0 \qquad \forall t$$

$$A + D = 0, \quad 2C - A = 0, \quad B = 0 \quad \text{tj.}$$

$$D = -A, \quad C = \frac{A}{2}, \quad B = 0$$

$$Ax + \frac{A}{2}z - A = 0 \qquad /:A$$

$$x + \frac{1}{2}z - 1 = 0$$

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = 0$$ jedn. osk. ravni u $M(x_1, y_1, z_1)$

$\vec{t} \perp \vec{b}$

$\vec{t} \perp \vec{n}$

$\vec{b} \perp \vec{n}$

$\dot{x}(x-x_1) + \dot{y}(y-y_1) + \dot{z}(z-z_1) = 0$ jedn. norm. ravni

$\ddot{x}(x-x_1) + \ddot{y}(y-y_1) + \ddot{z}(z-z_1) = 0$ jedn. rektif. ravni

VI GLAVA

DIFERENCIJALNA GEOMETRIJA

§ 1. Krive u prostoru

Ako je u trodimenzionalnom euklidskom prostoru kriva zadata jednačinom

$$\vec{r} = \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} \,,$$

ili u parametarskom obliku

$$x = x(t) \,, \quad y = y(t), \quad z = z(t) \,,$$

tada je dužina luka krive od tačke sa parametrom t_o *do tačke sa parametrom* t_1 *data formulom*

$$s = \int_{t_o}^{t_1} ds = \int_{t_o}^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt.$$

Vektor tangente, binormale i normale krive odredjujemo formulama

$$\vec{t} = \dot{\vec{r}}, \quad \vec{b} = \dot{\vec{r}} \times \ddot{\vec{r}}, \quad \vec{n} = \vec{b} \times \vec{t} \quad \left(\dot{\vec{r}} = \frac{d\vec{r}}{dt}\,, \ \ddot{\vec{r}} = \frac{d^2\vec{r}}{dt^2}, \ \dddot{\vec{r}} = \frac{d^3\vec{r}}{dt^3}\right)$$

i njihove ortove ćemo obeležiti respektivno sa

$$\vec{t}_o, \ \vec{b}_o, \ \vec{n}_o.$$